# Unit Disk Cover Problem

Rashmisnata Acharyya<sup>1</sup>, Manjanna B<sup>2</sup> and Gautam K. Das<sup>2</sup>

Department of Computer Science and Engineering,
Tezpur University, Assam, 786028, India
Department of Mathematics,
Indian Institute of Technology, Guwahatii, 781 - 039, India

**Abstract.** Given a set  $\mathcal{D}$  of unit disks in the Euclidean plane, we consider (i) the discrete unit disk cover (DUDC) problem and (ii) the rectangular region cover (RRC) problem. In the DUDC problem, for a given set  $\mathcal{P}$  of points the objective is to select minimum cardinality subset  $\mathcal{D}^* \subseteq \mathcal{D}$  such that each point in  $\mathcal{P}$  is covered by at least one disk in  $\mathcal{D}^*$ . On the other hand, in the RRC problem the objective is to select minimum cardinality subset  $\mathcal{D}^{**} \subseteq \mathcal{D}$  such that each point of a given rectangular region  $\mathcal{R}$  is covered by a disk in  $\mathcal{D}^{**}$ . For the DUDC problem, we propose an  $(9+\epsilon)$ -factor  $(0 < \epsilon \le 6)$  approximation algorithm. The previous best known approximation factor was 15 [12]. For the RRC problem, we propose (i) an  $(9+\epsilon)$ -factor  $(0 < \epsilon \le 6)$  approximation algorithm, (ii) an 2.25-factor approximation algorithm in reduce radius setup, improving previous 4-factor approximation result in the same setup [11].

The solution of DUDC problem is based on a PTAS for the subproblem LSDUDC, where all the points in  $\mathcal{P}$  are on one side of a line and covered by the disks centered on the other side of that line.

## 1 Introduction

In the unit disk cover (UDC) problem, we consider two problems, namely the discrete unit disk cover (DUDC) problem and the rectangular region cover (RRC) problem. In the DUDC problem, given a set  $\mathcal{P} = \{p_1, p_2, \dots, p_n\}$  of n points and a set  $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$  of m unit disks in the Euclidean plane, we wish to determine the minimum cardinality set  $\mathcal{D}^* \subseteq \mathcal{D}$  such that  $\mathcal{P} \cap \mathcal{D}^* = \mathcal{P}$ . In the rectangular region cover (RRC) problem, given a rectangular region  $\mathcal{R}$  and a set  $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$  of m unit disks in the Euclidean plane, the objective is to determine the minimum cardinality set  $\mathcal{D}^{**} \subseteq \mathcal{D}$  such that  $\mathcal{R} \cap \mathcal{D}^{**} = \mathcal{R}$ . The DUDC and RRC problems are a geometric version of the general set cover problem which is known to be NP-complete [14]. The general set cover problem is not approximable within  $c \log n$ , for some constant c, where n is the size of input. Unfortunately, both the DUDC and RRC problems are also NP-complete [14], but unlike general set cover problem, DUDC and RRC problems admit a constant factor approximation results. These two problems have been studied extensively due to their wide applications in wireless networks [6, 11, 19].

#### 1.1 Related work

The DUDC problem has a long history in the literature. It is a NP-complete problem [14]. The first constant factor approximation algorithm has been proposed by Brönnimann and Goodrich [4]. Their algorithm is based on the epsilon nets concept. After that many authors proposed constant factor approximation algorithm for the DUDC problem [3, 6–8, 17, 18]. A summary of such results are available in [10]. Using local search, Mustafa and Ray [17] proposed a PTAS for the DUDC problem. The time complexity of their PTAS is  $O(m^{2 \cdot (\frac{8\sqrt{2}}{\epsilon})^2 + 1}n)$ for  $0 < \epsilon \le 2$ . Therefore, the fastest operation of this algorithm is obtained when  $\epsilon = 2$  for a 3-factor approximation result in  $O(m^{65}n)$  time, which is not practical even for m=2. This leads to further research on the DUDC problem for finding constant factor approximation algorithm with reasonable running time. Das et al. [10] proposed an 18-factor approximation algorithm. The running time of their algorithm is  $O(mn + n \log n + m \log m)$ . Recently, Fraser and López-Ortiz [12] proposed an 15-factor approximation algorithm for the DUDC problem, which runs in  $O(m^6n)$  time. Das et al. [9] studied a restricted version of the DUDC problem, where all the centers of disks in  $\mathcal{D}$  are within a unit disk and all the points in  $\mathcal{P}$  are outside of that unit disk. They proposed an 2-factor approximation algorithm for this restricted version of the DUDC problem, which runs in  $O((m+n)^2)$  time.

In the way to solve DUDC problem, some authors consider a restricted version of the DUDC problem. In the literature it is known as line-separable discrete unit disk cover (LSDUDC) problem [6]. In this problem, the plane being divided into two half-planes  $\ell^+$  and  $\ell^-$  defined by a line  $\ell$ , all the points in  $\mathcal{P}$  are in  $\ell^+$  and the centers of disks in  $\mathcal{D}$  are in  $\ell^+$  outh that each point in  $\mathcal{P}$  is covered by at least one disk centered in  $\ell^-$ . Carmi et al. [8] described an 4-factor approximation algorithm for the LSDUDC problem. Latter, Claude et al. [6] proposed an 2-factor approximation algorithm for LSDUDC problem. Another restricted version of the DUDC problem is within strip discrete unit disk cover (WSDUDC) problem. In this problem, all the points in  $\mathcal{P}$  and center of the disks in  $\mathcal{D}$  are within a strip of width h. Das et al. [10] proposed an 6-factor approximation algorithm for  $h = 1/\sqrt{2}$ . Latter, Fraser and López-Ortiz [12] proposed an  $3\lceil 1/\sqrt{1-h^2} \rceil$ -factor approximation result for  $0 \le h < 1$ . They also proposed an 3-factor (resp. 4-factor) algorithm for  $h \le 4/5$  (resp.  $h \le 2\sqrt{2}/3$ ).

Agarwal and Sharir [2] studied Euclidean k-center problem. In this problem, a set  $\mathcal{P}$  of n points, a set  $\mathcal{O}$  of m points, and an integer k are given. The objective is to find k disks centered at the points in  $\mathcal{O}$  such that each point in  $\mathcal{P}$  is covered by at least one disk and the radius of largest disk is minimum. This problem is known to be NP-hard [2]. For fixed k, Hwang et al. [15] presented a  $m^{O(\sqrt{k})}$ -time algorithm. Latter, Agarwal and Procopiuc [1] presented  $m^{O(k^{1-1/d})}$ -time algorithm for the d-dimensional points. Fowler et al. [13] considered the minimum geometric disk cover problem, where input is a set  $\mathcal{P}$  of points in the Euclidean plane and the objective is to compute minimum cardinality set  $\mathcal{X}$  of unit disks such that each point in  $\mathcal{P}$  is covered by at least one disks in  $\mathcal{X}$ . They proved

that the problem is NP-hard. Hochbaum and Maass [16] proposed a PTAS for the geometric disk cover problem.

A sector is a maximal region formed by the intersection of a set of disks such that all points within the sector are covered by the same set of disks. Funke et al. [11] proposed greedy sector cover algorithm for RRC problem. The approximation factor of this algorithm is  $O(\log w)$ , where w is the maximum number of sectors covered by a single disk. They proved that the greedy sector cover algorithm has an approximation algorithm no better than  $\Omega(\log w)$ . In the same paper, they proposed grid placement algorithm (based on the algorithm proposed by Bose et al. [5]) and proved that their algorithm produce  $18\pi$ -factor approximation result. Though the algorithm is not guaranteeing full coverage of the region of interest, the area that remains uncovered can be bounded by the number of chosen grids. In the same paper, they have also considered RRC problem in different setup. We denote this setup as reduced radius setup. In the reduce radius setup, the region of interest  $\mathcal{R}$  is also covered by the disks in  $\mathcal{D}$ after reducing their radius to  $(1 - \delta)$ .  $\delta$  is said to be reduce radius parameter. The reduce radius setup has many applications in wireless sensor networks, where coverage remains stable under small perturbations of sensing ranges/positions. In this setup an algorithm  $\mathcal{A}$  is said to be  $\beta$ -factor approximation if  $\frac{|\overline{\mathcal{A}}_{out}|}{|opt|} \leq \beta$ , where  $\mathcal{A}_{out}$  is the output of algorithm  $\mathcal{A}$  and opt is the optimum set of disks with reduced radius covering the region of interest. In reduce radius setup, Funke et al. [11] proposed an 4-factor approximation algorithm for RRC problem.

#### 1.2 Our Results

We provide a PTAS (i.e.,  $(1+\mu)$ -factor approximation algorithm) for LSDUDC problem, which runs in  $O(m^{2(1+\frac{1}{\mu})}n)$  time  $(0<\mu\leq 1)$ . Using this PTAS, we present an  $(9+\epsilon)$ -factor approximation algorithm for DUDC problem in  $O(\max(m^{2(1+\frac{6}{\epsilon})}n,m^6n)$  time, where  $0<\epsilon\leq 6$ . For the RRC problem, we describe an  $(9+\epsilon)$ -factor approximation algorithm using the algorithm for DUDC. We also propose an 2.25-factor approximation algorithm in the reduce radius setup. The previous best known approximation factor was 4 [11].

## 2 PTAS for LSDUDC Problem

Let  $\ell$  be a horizontal line. We use  $\ell^+$  (resp.  $\ell^-$ ) to denote the half-plane above (resp. below) the line  $\ell$ . For LSDUDC problem, we assume that each point in  $\mathcal{P}$  is in  $\ell^-$  and center of each disk in  $\mathcal{D}$  is in  $\ell^+ \cup \ell^-$  such that union of the disks centered in  $\ell^+$  covers all points in  $\mathcal{P}$ . The objective is to find minimum cardinality set  $\mathcal{D}^* \subseteq \mathcal{D}$  such that each point in  $\mathcal{P}$  is covered by at least one disk in  $\mathcal{D}^*$ . Now, we want to define some terminology.

**Definition 1.** We use  $\mathcal{U}$  (resp.  $\mathcal{L}$ ) to denote the set of disks in  $\mathcal{D}$  with centers in  $\ell^+$  (resp.  $\ell^-$ ). We use circ(d) (resp.  $\alpha(d)$ ) to denote the boundary arc (resp. center) of the disk d. A disk  $d \in \mathcal{U}$  is said to be lower boundary disk if there does

not exit  $X \subseteq \mathcal{U} \setminus \{d\}$  such that  $d \cap \ell^- \subset (\cup_{D \in X} D) \cap \ell^-$ . For a lower boundary disk  $d \in \mathcal{U}$ , we use the term lower region to denote the region  $d \cap \ell^-$  and lower arc to denote the arc  $circ(d) \cap \ell^-$  (see Fig. 1). We use  $D_{\ell} = \{d^1, d^2, \ldots, d^s\} \subseteq \mathcal{U}$  to denote the set of all lower boundary disks. We use  $B_{region}$  to denote the region covered by the disks in  $D_{\ell}$ .

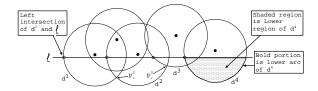


Fig. 1. Lower region, left intersection and lower boundary disks

Needless to mention that each disk in  $D_\ell$  intersect the horizontal line  $\ell$ . Without loss of generality assume that  $d^1, d^2, \ldots, d^s$  is the sorted order from left to right based on their left intersection point with the line  $\ell$  (see Fig. 1). Since centers of the disks in  $D_\ell$  are in  $\ell^+$  the number of intersection (if any) of two disks of  $D_\ell$  in  $\ell^-$  is one. For each disk  $d^i \in D_\ell$  we define two points, namely  $p^i_l$  and  $p^i_r$  as follows:

- $p_l^i$ : If the disk  $d^i$  has intersection with  $d^{i-1}$  in  $\ell^-$ , then  $p_l^i$  is the intersection point between  $circ(d^{i-1})$  and  $circ(d^i)$  in  $\ell^-$ , otherwise  $p_l^i$  is the left intersection point between  $\ell$  and  $circ(d^i)$ .
- $p_r^i$ : If the disk  $d^i$  has intersection with  $d^{i+1}$  in  $\ell^-$ , then  $p_r^i$  is the intersection point between  $circ(d^{i+1})$  and  $circ(d^i)$  in  $\ell^-$ , otherwise  $p_r^i$  is the right intersection point between  $\ell$  and  $circ(d^i)$ .

Where  $d^0$  and  $d^{s+1}$  are the two dummy disks having no intersection with  $d^1$  and  $d^s$  respectively. For each  $i=1,2,\ldots,s$  let  $\mathcal{P}_i(\subseteq \mathcal{P})$  be the set of points lies between two vertical lines through  $p^i_l$  and  $p^i_r$ . Let  $e^i$  be the vertical line through the point  $p^i_r$  for  $i=1,2,\ldots,s$ . We use  $e^{i-}$  (resp.  $e^{i+}$ ) to denote the region in the left (resp. right) side of the vertical line  $e^i$ . Let  $D^{i-}$  (resp.  $D^{i+}$ ) be the optimum cover of the points in  $\mathcal{P} \cap e^{i-}$  (resp.  $\mathcal{P} \cap e^{i+}$ ).

**Observation 1** For two disks  $d', d'' \in \mathcal{D}$ , if  $d', d'' \in D^{i-}$  and  $d', d'' \in D^{i+}$ , then both the disks d' and d'' intersect  $e^i$  and circ(d') and circ(d'') intersect in  $B_{region}$ .

Proof. Both the disks d' and d'' intersect  $e^i$  because  $d', d'' \in D^{i-}$  and  $d', d'' \in D^{i+}$ . Now, if circ(d') and circ(d'') does not intersect in  $B_{region}$ , then either  $\mathcal{P} \cap d' \subseteq \mathcal{P} \cap d''$  or  $\mathcal{P} \cap d'' \subseteq \mathcal{P} \cap d'$ . Therefore, both d' and d'' can not appear in the solutions  $D^{i-}$  and  $D^{i+}$ .

## Algorithm 1 LSDUDC( $\mathcal{P}, \mathcal{D}, k, \ell$ )

- 1: **Input:** Set  $\mathcal{P}$  of points, set  $\mathcal{D}$  of unit disks, a positive integer k and a horizontal line  $\ell$  such that  $\mathcal{P} \cap \ell^- = \mathcal{P}$  and union of the disks centered in  $\ell^+$  covers all the points in  $\mathcal{P}$ .
- 2: **Output:** Set  $\mathcal{D}^* \subseteq \mathcal{D}$  of disks covering all the points in  $\mathcal{P}$ .
- 3: Set  $\mathcal{D}^* \leftarrow \emptyset$
- 4: Find lower boundary disks set  $D_{\ell}$  and arrange them from left to right as defined above. Let  $D_{\ell} = \{d^1, d^2, \dots, d^s\}$  be the lower boundary disks from left to right.
- 5: **for**  $(i = 1, 2, \dots s)$  **do**
- 6: Compute the set  $\mathcal{P}_i(\subseteq \mathcal{P})$
- 7: end for
- 8: Set  $i \leftarrow 1$
- 9: while  $(i \le s)$  do
- 10: Find the maximum index j such that  $\bigcup_{\theta=i,i+1,...,j} \mathcal{P}_{\theta}$  is covered by a set  $\mathcal{D}_1(\subseteq \mathcal{D})$  of size k.
- 11:  $\mathcal{D}^* = \mathcal{D}^* \cup \mathcal{D}_1, i \leftarrow j+1$
- 12: end while
- 13: Return  $\mathcal{D}^*$

**Definition 2.** A pair (d', d'') of disks is said to be week (resp. strong) cover pair if circ(d') and circ(d'') intersect once (resp. twice) in  $B_{region}$ .

**Lemma 1.** For a week cover pair (d', d'');  $d', d'' \in D^{i-}$  and  $d', d'' \in D^{i+}$  can not happen simultaneously.

Proof. On contrary, assume  $d', d'' \in D^{i-}$  and  $d', d'' \in D^{i+}$ . By the definition of week cover pair, circ(d') and circ(d'') does not intersect either in  $e^{i-}$  or  $e^{i+}$ . Therefore, either (i)  $\mathcal{P}_{e^{i-}} \cap d' \subset \mathcal{P}_{e^{i-}} \cap d''$  or  $\mathcal{P}_{e^{i-}} \cap d'' \subset \mathcal{P}_{e^{i-}} \cap d'$  or (ii)  $\mathcal{P}_{e^{i+}} \cap d' \subset \mathcal{P}_{e^{i+}} \cap d'' \subset \mathcal{P}_{e^{i+}} \cap d'$ , where  $\mathcal{P}_{e^{i-}}$  (resp.  $\mathcal{P}_{e^{i+}}$ ) is the set of points in  $\mathcal{P}$  to the left (resp. right) of  $e^i$  (see Fig. 2). Thus, both the disks d', d'' can not be in  $D^{i-}$  and  $D^{i+}$ .

**Lemma 2.** For a strong cover pair (d', d''), if  $d', d'' \in D^{i-}$  and  $d', d'' \in D^{i+}$ , then one intersection between circ(d') and circ(d'') lies in  $e^{i-}$  and other intersection lies in  $e^{i+}$ .

*Proof.* If both the intersections between circ(d') and circ(d'') lies either in  $e^{i-}$  or  $e^{i+}$ , then from Observation 1 and the proof of Lemma 1  $d', d'' \in D^{i-}$  and  $d', d'' \in D^{i+}$  can not happen simultaneously, which is a contradiction. Thus, the lemma follows.

**Lemma 3.** For a strong cover pair (d', d'') with  $\alpha(d')$  is above  $\alpha(d'')$ , if the intersection of circ(d') and circ(d'') lies within a lower boundary disk d and both  $\alpha(d')$  and  $\alpha(d'')$  lies either right of  $\alpha(d)$  or left of  $\alpha(d)$ , then one intersection of circ(d) and circ(d') happen above the horizontal line  $\ell$ .

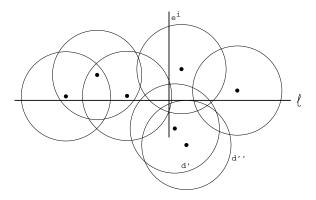


Fig. 2. Proof of Lemma 1

Proof. With out loss of generality assume that both  $\alpha(d')$  and  $\alpha(d'')$  lie to the right of  $\alpha(d)$ . Let a and b be the two intersection points of circ(d) and circ(d'). Since  $\alpha(d')$  is above  $\alpha(d'')$ , the intersection of circ(d') and circ(d'') lies within d and both  $\alpha(d')$  and  $\alpha(d'')$  lie to the right of  $\alpha(d)$ ,  $\alpha(d')$  should lies above at least one point among a or b. Let  $\alpha(d')$  lies above a. By symmetry,  $\overline{\alpha(d')}$ , a and  $\overline{\alpha(d)}$ , b are parallel (see Fig. 3). Thus, b must be above  $\alpha(d)$  i.e., b must be above  $\ell$ .

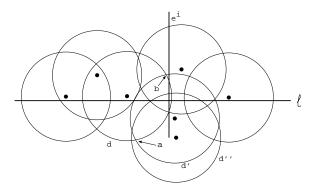


Fig. 3. Proof of Lemma 3

# **Lemma 4.** $|D^{i-} \cap D^{i+}| \leq 2$ .

*Proof.* On contrary, assume that  $d_x, d_y, d_z \in D^{i-} \cap D^{i+}$ . Since  $d_x, d_y, d_z \in D^{i-}$  as well as  $d_x, d_y, d_z \in D^{i+}$ , the disks  $d_x, d_y, d_z$  intersect each other in  $B_{region}$ . If any pair  $(d, d') \in \Gamma = \{(d_x, d_y), (d_x, d_z), (d_y, d_z)\}$  do not form a week cover pair

nor strong cover pair, then either  $d \cap B_{region} \subseteq d' \cap B_{region}$  or  $d \cap B_{region} \supseteq d' \cap B_{region}$ , which contradict the fact that  $d_x, d_y, d_z \in D^{i-} \cap D^{i+}$ . Again, from Lemma 1, no pair in  $\Gamma$  form a week cover pair because  $d_x, d_y, d_z \in D^{i-}$  as well as  $d_x, d_y, d_z \in D^{i+}$ . Therefore, each pair in  $\Gamma$  form a strong cover pair. Without loss of generality assume that  $\alpha(d_x)$  is below  $\alpha(d_y)$  and  $\alpha(d_y)$  is below  $\alpha(d_z)$  (see Fig. 4). If a is the intersection between  $circ(d_x)$  and  $circ(d_y)$  inside the lower boundary disk d (say) and below the horizontal line  $\ell$ , then one intersection between  $circ(d_y)$  and  $circ(d_z)$  lies inside of d (from Lemma 3). Therefore,  $(d_x \cup d_y \cup d_z) \cap \mathcal{P} \cap e^{i-} \subseteq (d_x \cup d) \cap \mathcal{P} \cap e^{i-}$ , which implies that  $D^{i-}$  is not optimum, leading to a contradiction. Thus, the lemma follows.

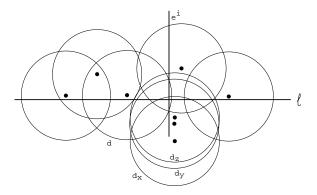


Fig. 4. Proof of Lemma 4

The following theorem says that the LSDUDC problem admits a PTAS.

**Theorem 1.** Algorithm 1 produce  $(1 + \frac{2}{k-2})$ -factor approximation results in  $O(m^k n)$  time.

Proof. For some integer t, let  $j_1, j_2, \ldots, j_t$  the values of j in the while loop (line number 9) of the Algorithm 1. Let  $\mathcal{Q}_v = \bigcup_{i=j_{v-1}+1, j_{v-1}+2, \ldots, j_v} \mathcal{P}_i$  for  $v = 1, 2, \ldots, t$ , where  $j_0 = 0$ . Algorithm 1 finds a covering for the sets  $\{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_t\}$  independently with each of size k (optimum size because in each iteration of the while loop in line number 9, Algorithm 1 finds maximum value of j's) except the covering of  $\mathcal{Q}_t$ . Let  $\mathcal{D}^1, \mathcal{D}^2, \ldots, \mathcal{D}^t$  be the covering for  $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_t$  respectively. Lemma 4 says that  $\mathcal{D}^i \cap \mathcal{D}^{i+1} \leq 2$ . Therefore, the total number of disks required to cover all the points by Algorithm 1 is  $k(t-1) + |\mathcal{D}^t|$  whereas at least  $k(t-1) + |\mathcal{D}^t|$  disks required in the optimum solution. Thus, the approximation factor of the Algorithm 1 is  $k(t-1) + \frac{1}{k-2}$ .

The execution time to find lower boundary disks and arrange them from left to right (line number 4) is  $O(m \log m)$ , where  $m = |\mathcal{D}|$ . To compute  $\mathcal{P}_i$  for i = 1, 2, ...s (for loop in line number 5)  $O(n \log n)$  time is required. The time

complexity of the while loop (line number 9) is  $O(m^k n)$ . Thus, the total time complexity of the Algorithm 1 is  $O(m^k n)$ .

## 2.1 An $(9 + \epsilon)$ -factor Approximation Algorithm for DUDC Problem

In this section, we wish to describe an  $(9+\epsilon)$ -factor approximation algorithm for DUDC Problem, where a set  $\mathcal P$  of n points and a set  $\mathcal D$  of m unit disks are distributed in the Euclidean plane and objective is to choose minimum cardinality set  $\mathcal D^*(\subseteq \mathcal D)$  such that union of the disks in  $\mathcal D^*$  covers  $\mathcal P$ . From Theorem 1, LSDUDC problem has an  $(1+\mu)$ -factor approximation algorithm  $(\mu=\frac{2}{k-2})$  and the running time of the algorithm is  $O(m^{2(1+\frac{1}{\mu})}n)$ . Das et al. [10] proved that any instance of DUDC problem can be partitioned into several instances of LSDUDC and WSDUDC (with strip width  $1/\sqrt{2}$ ) problems. They also proved that the approximation factor of the DUDC problem is

 $6 \times \text{(approximation factor of LSDUDC problem)} + \text{approximation factor of WSDUDC (width } \geq 1/\sqrt{2}) \text{ problem.}$ 

Fraser and López-Ortiz [12] proposed an 3-factor approximation algorithm for WSDUDC (with width  $h \leq 4/5$ ) problem. Therefore, we have the following theorem for DUDC problem.

**Theorem 2.** The DUDC problem admits  $(9+\epsilon)$ -factor approximation result in  $O(\max(m^6n, m^{2(1+\frac{6}{\epsilon})}n))$  time.

*Proof.* The approximation factor of the DUDC problem is  $(6 \times (approximation factor of LSDUDC problem) + approximation factor of WSDUDC (width <math>\geq 1/\sqrt{2}$ ) problem) [10]. Therefore, the approximation factor for the DUDC problem is  $6 \times (1 + \mu) + 3 = 9 + \epsilon$ , where  $\epsilon = 6\mu$ . The time complexity result follows from (i) time complexity of WSDUDC is  $O(m^6n)$  [12] and (ii) above discussion.

**Corollary 1.** The DUDC problem has an 12-factor approximation result with time complexity equal to the 3-factor approximation algorithm for WSDUDC problem.

*Proof.* The time complexity of 3-factor approximation algorithm for the WS-DUDC problem is  $O(m^6n)$  [12]. If we set  $\epsilon = 3$ , then approximation factor of the DUDC problem is 12 (see Theorem 2) and the running time of the algorithm is  $O(m^6n)$  (see Theorem 2). Thus, the corollary follows.

## 3 Approximation Algorithms for RRC Problem

In the RRC problem, the inputs are (i) a set  $\mathcal{D}$  of m unit disks and (ii) a rectangular region  $\mathcal{R}$  and the objective is to choose minimum cardinality set  $\mathcal{D}^{**}(\subseteq \mathcal{D})$  such that each point in  $\mathcal{R}$  is covered by at least one disk in  $\mathcal{D}^{**}$ . In this problem, we assume that  $\mathcal{R}$  is covered by the union of the disks in  $\mathcal{D}$ , otherwise the RRC problem has no feasible solution. A sector f inside  $\mathcal{R}$  is a maximal region inside  $\mathcal{R}$  formed by the intersection of a set of disks such that

each point within f is covered by the same set of disks. Let  $\mathcal{F}$  be the set of all sectors (inside  $\mathcal{R}$ ) formed by  $\mathcal{D}$ . Therefore, the size of  $\mathcal{F}$  is at most  $O(m^2)$ . Now we construct a set of points  $\mathcal{T}$  as follows: for each sector  $f \in \mathcal{F}$  we add one arbitrary point  $p \in f$  to  $\mathcal{T}$ . Thus, we have the following theorem:

**Theorem 3.** The RRC problem has  $(9+\epsilon)$ -factor approximation algorithm with running time  $O(\max(m^8, m^{4(1+3/\epsilon)}))$ .

*Proof.* Consider an arbitrary point  $p \in \mathcal{T}$ . Let  $f \in \mathcal{F}$  be the sector in which the point p lies. From the definition of sector, if a disk  $d \in \mathcal{D}$  covers p, then the disk d also covers the whole sector f. Therefore, the instance  $(\mathcal{R}, \mathcal{D})$  of the RRC problem is exactly same as the instance  $(\mathcal{T}, \mathcal{D})$  of the DUDC problem. Note that  $\mathcal{T} = O(m^2)$ . Thus, the theorem follows from Theorem 2 by putting  $n = m^2$ .  $\square$ 

### 3.1 RRC Problem in Reduce Radius Setup

In this subsection we consider the RRC problem in reduce radius setup. For a given set  $\mathcal{D}$  of unit disks and a rectangular region  $\mathcal{R}$  such that  $\mathcal{R}$  is covered by the disks in  $\mathcal{D}$  after reducing their radius to  $(1-\delta)$ , the objective is to choose minimum cardinality set  $\mathcal{D}^{**}(\subseteq \mathcal{D})$  whose union covers  $\mathcal{R}$ . In the reduce radius setup an algorithm  $\mathcal{A}$  is said to be  $\beta$ -factor approximation if  $\frac{|\mathcal{A}_{out}|}{|opt|} \le \beta$ , where  $\mathcal{A}_{out}$  is the output of  $\mathcal{A}$  and opt is the optimum set of disks with reduced radius covering the region of interest. The reduce radius setup has many applications in wireless sensor networks, where coverage remains stable under small perturbations of sensing ranges and their positions. Here we propose an 2.25-factor approximation algorithm for this problem. The best known approximation factor for the same problem was 4 [11].

**Observation 2** Let  $\delta = \nu/\sqrt{2}$  and d be an unit disk centered at a point p. If d' is a disk of radius  $(1 - \delta)$  centered within a square of size  $\nu \times \nu$  centered at p, then  $d' \subseteq d$ .

*Proof.* The observation follows from the fact that the maximum distance of any point within the square of size  $\nu \times \nu$  from the center point p is  $\nu/\sqrt{2} (=\delta)$ .  $\square$ 

Consider a grid with cells of size  $\nu \times \nu$  over the region  $\mathcal{R}$ . Like Funke et al. [11] we also snap the center of each  $d \in \mathcal{D}$  to the closest vertex of the grid and set its radius to  $(1-\delta)$ . Let  $\mathcal{D}'$  be the set of disks with radius  $(1-\delta)$  after snapping their centers. Let  $\mathcal{R}'$  be a square of size  $4 \times 4$  on the Euclidean plane contained in  $\mathcal{R}$ . We define the regions TOP, DOWN, LEFT, RIGHT, TOP-LEFT, TOP-RIGHT, DOWN-LEFT, DOWN-RIGHT as shown in Fig. 5. We now construct a set  $\mathcal{D}_{RS}(\subseteq \mathcal{D}')$  such that no disk  $d \notin \mathcal{D}_{RS}$  can participate to the optimum solution for covering the region  $\mathcal{R}'$  by  $\mathcal{D}'$ . The pseudo code for construction of  $\mathcal{D}_{RS}$  is given in Algorithm 2.

**Definition 3.** A disk  $d \in \mathcal{D}'$  dominates another disk  $d' \in \mathcal{D}'$  with respect to the region  $\mathcal{R}'$  if  $d \cap \mathcal{R}' \supset d' \cap \mathcal{R}'$ .

TOP-LEFT	TOP	TOP-RIGHT
LEFT	R'	RIGHT
DOWN-LEFT	DOWN	DOWN-RIGHT
P		

Fig. 5. Definition of different regions

## Algorithm 2 Algorithm $\mathcal{D}_{RS}(\mathcal{D}', \mathcal{R}', \nu)$

- 1: Input: Set  $\mathcal{D}'$  of disks, a square region  $\mathcal{R}'$  of size  $4 \times 4$  and grid size  $\nu$ .
- 2: **Output:**  $\mathcal{D}_{RS}(\subseteq \mathcal{D}')$  such that no disk  $d \notin \mathcal{D}_{RS}$  can participate to the optimum solution for covering the region  $\mathcal{R}'$  by  $\mathcal{D}'$ .
- 3: Set  $\mathcal{D}_{RS} \leftarrow \emptyset, \mathcal{D}_t \leftarrow \emptyset$
- 4: For each disk  $d \in \mathcal{D}'$  having center in  $\mathcal{R}'$ ,  $\mathcal{D}_{RS} = \mathcal{D}_{RS} \cup \{d\}$
- 5: For each horizontal grid line segment h in LEFT add a disk  $d \in \mathcal{D}'$  to  $\mathcal{D}_{RS}$  if (i)  $d \cap \mathcal{R}' \neq \emptyset$ , (ii) center of d lies on h and (iii) center of d closest to  $\mathcal{R}'$  than other disks having center on h. Similarly add disks to  $\mathcal{D}_{RS}$  for the regions RIGHT, TOP and DOWN.
- 6: for (each horizontal grid line segment h in TOP-RIGHT from bottom to top) do
- 7: Add a disk  $d \in \mathcal{D}'$  to  $\mathcal{D}_t$  if (i)  $d \cap \mathcal{R}' \neq \emptyset$ , (ii) center of d lies on h and (iii) there does not exits any disk  $d' \in \mathcal{D}_t$  dominating d.
- 8: end for
- 9:  $\mathcal{D}_{RS} = \mathcal{D}_{RS} \cup \mathcal{D}_t$
- 10: repeat steps 6-9 for TOP-LEFT, DOWN-LEFT and DOWN-RIGHT.
- 11: Return  $\mathcal{D}_{RS}$

**Lemma 5.** If  $d \in \mathcal{D}'$  and  $d \notin \mathcal{D}_{RS}$ , then d can not participate to the optimum solution for covering  $\mathcal{R}'$  by minimum number of disks in  $\mathcal{D}'$ .

Proof. Since  $d \notin \mathcal{D}_{RS}$  the center of d is in outside of  $\mathcal{R}'$  (see line number 4 of Algorithm 2). With out loss of generality assume that center of d is in LEFT and on the horizontal grid line segment h. By our construction of the set  $\mathcal{D}_{RS}$ , there exists a disk  $d' \in \mathcal{D}_{RS}$  centered on h such that (a)  $d' \cap \mathcal{R}' \neq \emptyset$ , (b) center of d' lies on h and (c) center of d' closest to  $\mathcal{R}'$  than other disks having center on h. Therefore, d' dominates d. Similarly, we can prove for other cases also. Thus, the lemma follows.

# **Lemma 6.** $|\mathcal{D}_{RS}| \leq \frac{16}{\nu^2} + \frac{20}{\nu}$ .

Proof. The lemma follows from the following facts: (i) the maximum number of grid vertices in  $\mathcal{R}'$  is  $\frac{16}{\nu^2}$  and each of them can contribute one disk in  $\mathcal{D}_{RS}$ , (ii) the maximum number of horizontal grid line segment in the regions TOP-LEFT, LEFT, DOWN-LEFT, DOWN-RIGHT, RIGHT and TOP-RIGHT that can contribute a disk in  $\mathcal{D}_{RS}$  is  $\frac{12}{\nu}$  and (iii) the maximum number of vertical grid line segment in the regions TOP and DOWN that can contribute a disk in  $\mathcal{D}_{RS}$  is  $\frac{8}{\nu}$ . Thus, the lemma follows.

From Observation 2 and Lemma 6, we can compute a cover of  $\mathcal{R}'$  by  $\mathcal{D}''(\subseteq \mathcal{D})$  with minimum number of disks using brute-force method, where  $\mathcal{D}''$  is the set

of disks of unit radius corresponding to the disks in  $\mathcal{D}_{RS}$ . The running time of the brute-force algorithm is  $O(2^{\frac{16}{\nu^2} + \frac{20}{\nu}})$  (see Lemma 6). Though, the worst-case running time of this brute-force algorithm is exponential in  $\frac{1}{\nu^2}$ , in practice, it is very small. Note that time complexity of our proposed algorithm is less than that of 4-factor approximation algorithm proposed by Funke et al. [11]. We now describe approximation factor of our propose algorithm for RRC problem in reduce radius setup.

**Theorem 4.** In the reduce radius setup, RRC problem has an 2.25-factor approximation algorithm.

*Proof.* From the above discussion, for rectangle of size  $4 \times 4$ , we have optimum solution for RRC problem. Note that the diameter of each disk of the RRC instance is 2. Therefore, we can apply shifting strategy described by Hochbaum and Maass [16] to solve RRC problem and the approximation factor is  $(1 + 1/2)^2 = 2.25$ . Thus, the theorem follows.

### 4 Conclusion

In this paper, we have proposed a PTAS for LSDUDC problem, improving previous 2-factor approximation result. Using this PTAS, we proposed an  $(9+\epsilon)$ -factor approximation algorithm for DUDC problem, improving previous 15-factor approximation result for the same problem. The running time of our proposed algorithm for  $\epsilon = 3$  (i.e., approximation factor of DUDC problem is 12) is same as the running time of 15-factor approximation algorithm. We have also proposed an  $(9 + \epsilon)$ -factor approximation algorithm for RRC problem, which runs in  $O(\max(m^8, m^{4(1+3/\epsilon)}))$  time. In the reduce radius setup, we proposed an 2.25factor approximation algorithm. The previous best known approximation factor was 4 [11]. The running time of our proposed algorithm for RRC problem in reduce radius setup is less than that of 4-factor approximation algorithm proposed in [11] for reasonably small values of  $\delta (=\frac{\nu}{\sqrt{2}})$ , where  $\delta$  is the radius reduction parameter. Since the number of disks participating in the solution of  $4 \times 4$  square is constant for fixed value of  $\delta$ , the number of disks participating in the solution of  $L \times L$  square is constant. Therefore, using the shifting strategy proposed by Hochbaum and W. Maass [16], we can design a PTAS for the RRC problem in reduce radius setup.

### References

- P. Agarwal and C. Procopiuc, Exact and approximation algorithms for clustering, Algorithmica, vol. 33, pp. 201 - 226, 2002.
- 2. P. Agarwal and M. Sharir, *Efficient algorithm for geometric optimization*, ACM Computing Surveys, vol. 30, pp. 412 458, 1998.
- 3. C. Ambühl, T. Erlebach, M. Mihalák, and M. Nunkesser, Constant factor approximation for minimum-weight (connected) dominating sets in unit disk graph,

- Proc. of the 9th International Conference on Approximation Algorithms for Combinatorial Optimization Problems, and 10th International Conference on Randomization and Computation, LNCS 4110, pp. 3 14, 2006.
- 4. H. Brönnimann and M. Goodrich, Almost optimal set covers in finite VC-dimension, Discrete and Computational Geometry, vol. 14, pp. 463 479, 1995.
- P. Bose, A. Maheshwari, P. Morin and J. Morrison, The grid placement problem, Proc. of the 7th International Workshop on Algorithms and Data Structures, LNCS -2125, pp. 180 - 191, 2001.
- F. Claude, G. K. Das, R. Dorrigiv, S. Durocher, R. Fraser, A. Lopez-Ortiz, B. G. Nickerson and A. Salinger, An improved line-separable algorithm for discrete unit disk cover, Discrete Mathematics, Algorithms, and Application, vol. 2, pp. 77 - 87, 2010.
- G. Călinescu, I. Măndoiu, P.J. Wan and A. Zelikovsky, Selecting forwarding neighbours in wireless ad hoc networks, Mobile Networks and Applications, vol. 9, pp. 101 - 111, 2004.
- 8. P. Carmi, M. J. Katz and N. Lev-Tov, Covering points by unit disks on fixed location, Proc. of the 18th International Symposium on Algorithms and Computation, LNCS 4835, pp. 644 655, 2007.
- 9. G. K. Das, S. Das and S. C. Nandy, *Homogeneous 2-hop broadcast in 2D*, Computational Geometry: Theory and Applications, vol. 43, pp. 182 190, 2010.
- G. K. Das, R. Fraser, A. Lopez-Ortiz and B. Nickerson, On the discrete unit disk cover problem, Proc. of the 5th International Workshop on Algorithms and Computation, LNCS - 6552, pp. 146 - 157, 2011.
- S. Funke, A. Kesseelman, F. Kuhn, Z. Lotker and M. Segal, Improved approximation algorithms for connected sensor cover, Wireless Networks, vol. 13, pp. 153-164, 2007.
- R. Fraser and A. López-Ortiz, The within-strip discrete unit disk cover problem, Proc. of the 24th Canadian Conference on Computational Geometry, pp. 61 - 66, 2012.
- 13. R. Fowler, M. Paterson and S. Tanimato, *Optimal packing and covering in the palne are NP-complete*, Information Processing Letters, vol. 12, pp. 133 137, 1981.
- M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman and Company, New York, 1979.
- 15. R. Hwang, R. Lee and R. Chang, The generalized searching over separators strategy to solve some NP-hard problems in subexponential time, Algorithmica, vol. 9, pp. 398 423, 1993.
- D. S. Hochbaum and W. Maass, Approximation schemes for covering and packing problems in image processing and VLSI, Journal of ACM, vol 32, pp. 130 - 136, 1085
- 17. N. H. Mustafa and S. Ray, Improved results on geometric hitting set problems, Discrete and Computational Geometry, vol. 44, pp. 883 895, 2010.
- 18. S. Narayanappa and P. Vojtechovsky, *An improved approximation factor for the unit disk covering problem*, Proc. of the 18th Canadian Conference on Computational Geometry, pp. 15 18, 2006.
- D. Yang, S. Misra, X. Fang, G. Xue and J. Zhang, Two-Tiered Constrained Relay Node Placement in Wireless Sensor Networks: Computational Complexity and Efficient Approximations, IEEE Transactions on Mobile Computing, vol. 11, pp. 1399 - 1411, 2012.